(S)PDEs in Geology

Insights from a Mathematician

Pathways into Mathematics of SPDEs, Heidelberg, March 11th 2026

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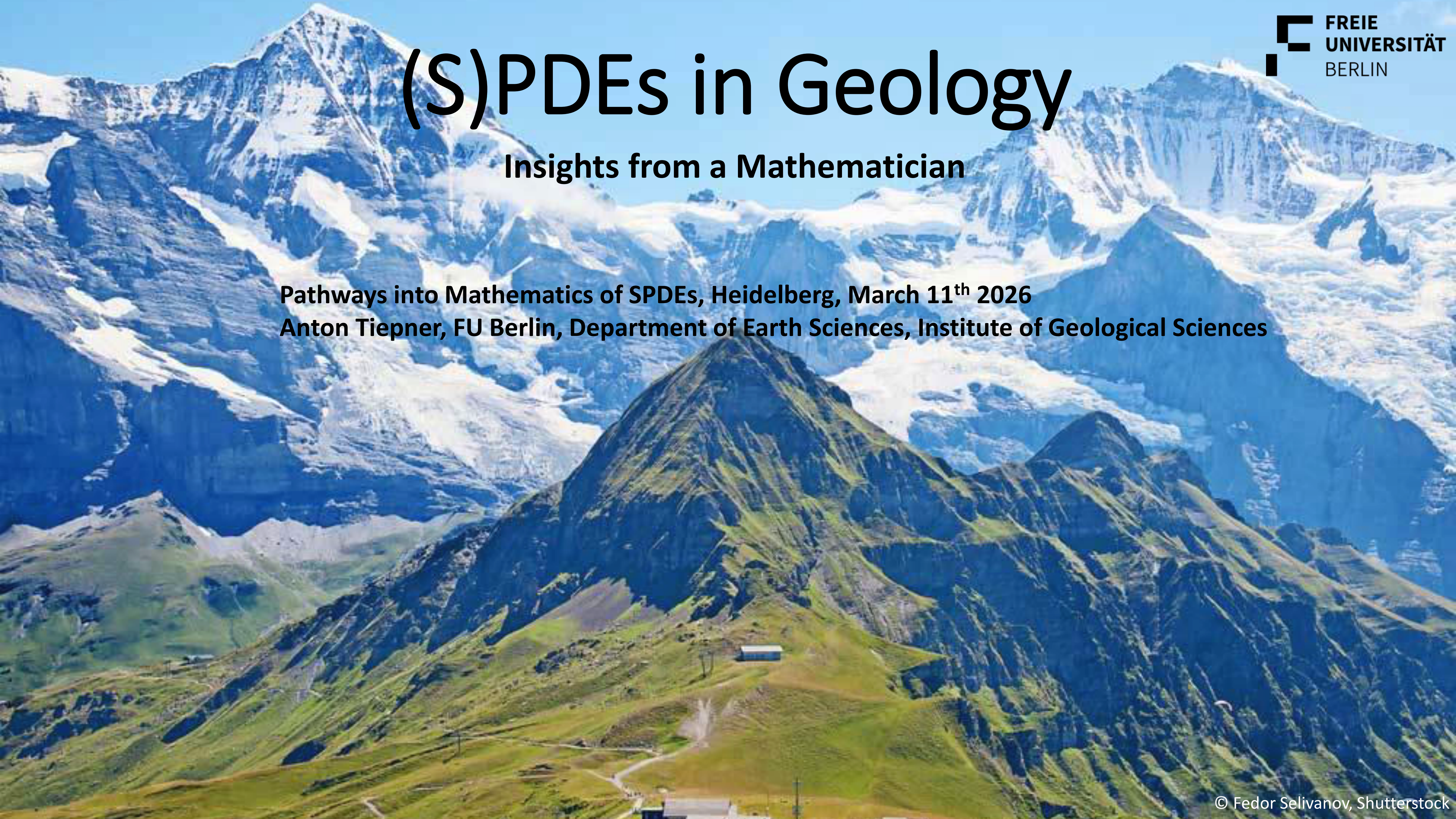

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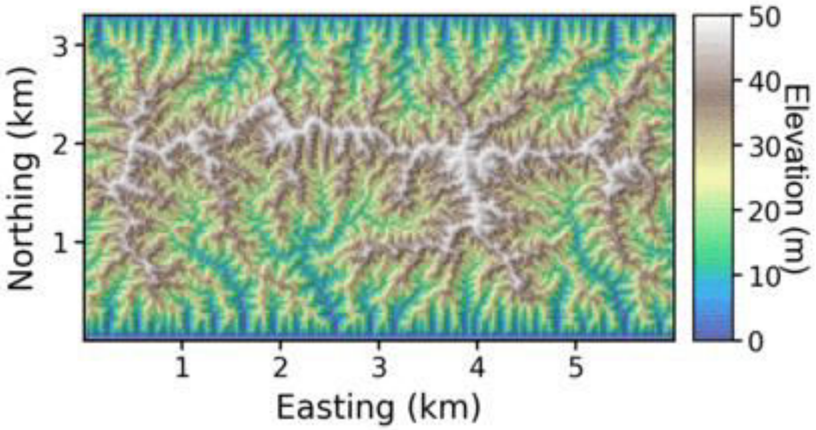
Landscape evolution models

What?

- Evolution of landscape through time
- Glacial, fluvial, marine, hillslopes…
- Tectonic, climatic, antropogenic
- Deterministic and stochastic components

How to find the governing equations?

- Physics
- Empirical observations
- Time series analysis/Dynamical system theory



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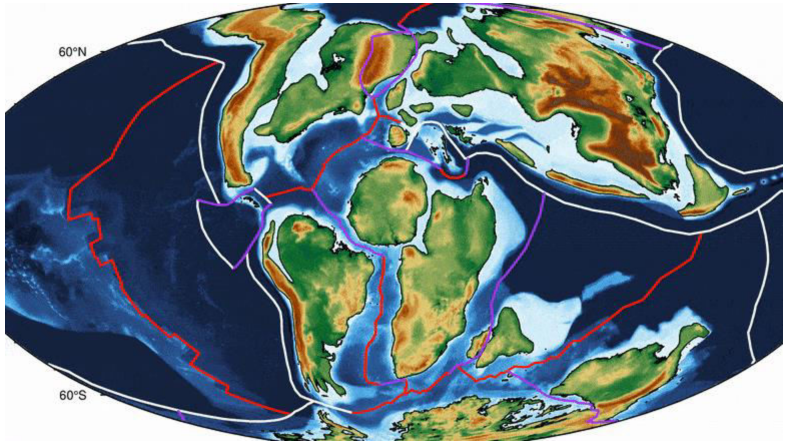


https://scitechdaily.com/100-million-years-unveiled-the-most-detailed-model-of-earths-surface-ever/

Landscape evolution models

https://blog.geogarage.com/2019/12/35-years-of-change-on-mamore-river.html

Why?

- Prediction
  - Storage of nuclear waste
  - Early-warning signals/tipping points
  - Ecological/economical safety
- Prevention

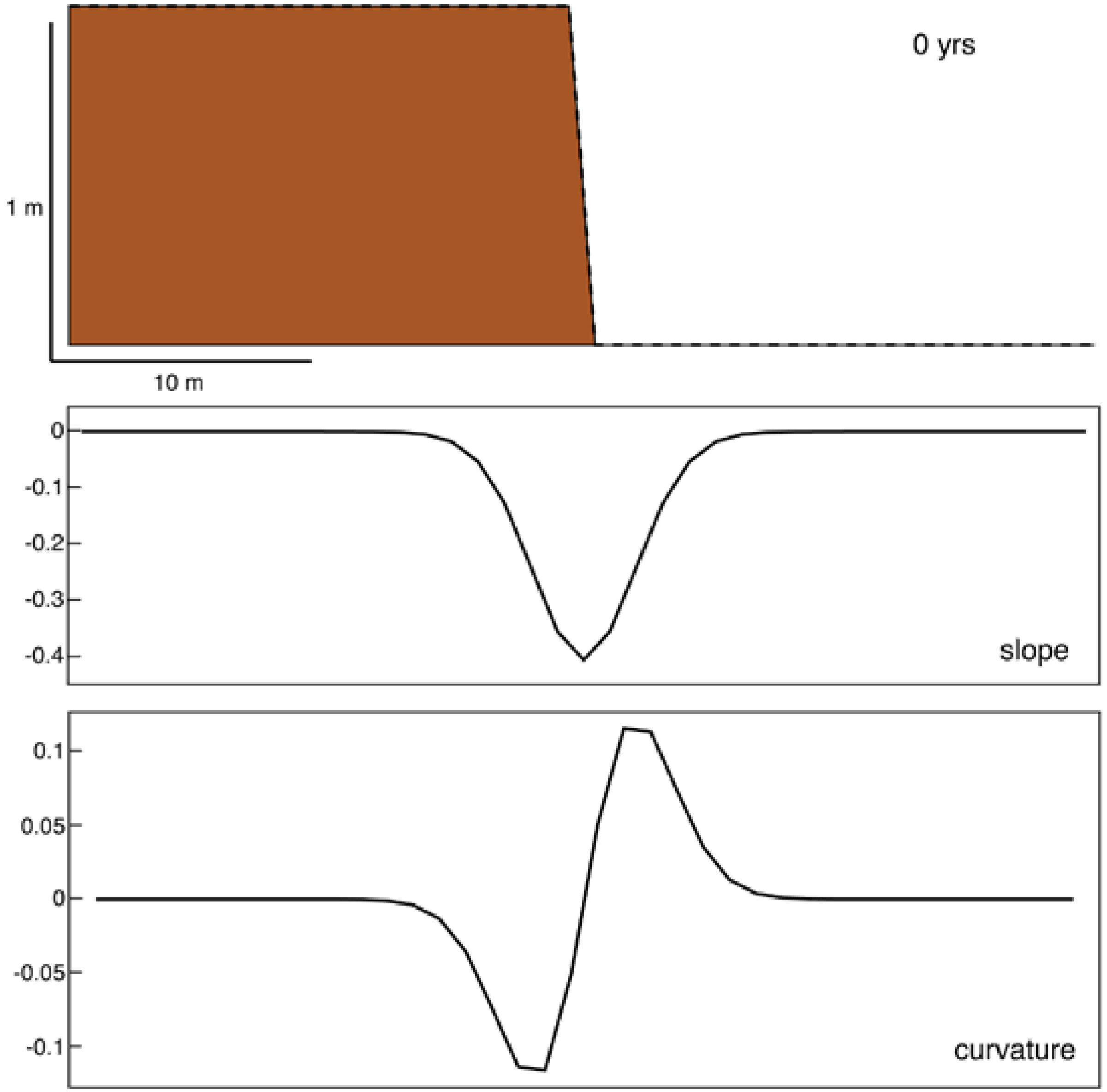
Timescales and spatial scales

- Event-based: Seconds to hours (flooding, landslides…), local
- Geological timescales: Decades to million years (River evolution, coastal erosion, orogeny…), regional/global

Hillslope diffusion

- Smoothing through soil creep, rain splash or tree throw
- **Conservation of mass**
  $$\frac{\partial h}{\partial t} = -\frac{1}{\rho}\frac{\partial q_s}{\partial x}$$
- **Slope-proportional transport**
  $$q_s = -\rho\kappa\frac{\partial h}{\partial x}$$
- **Diffusion**
  $$\frac{\partial h}{\partial t} = \kappa\frac{\partial^2 h}{\partial x^2}$$

Sediment flux per unit with $\boldsymbol{q_s}$, elevation $\boldsymbol{h}$, density $\boldsymbol{\rho}$ and sediment-transport diffusivity $\boldsymbol{\kappa}$



https://hinderedsettling.com/2009/11/26/hillslope-diffusion/

Hillslope diffusion

Uplift rate U is included to model uplift/subsidence. It can be viewed as either source or noise term: $\frac{\partial h}{\partial t} = \kappa \frac{\partial^2 h}{\partial x^2} + U(t, x)$.

Uplift rates are induced by e.g. tectonics, isostasy, compaction.

Slope-proportional transport, i.e. $q_s = -\rho\kappa \frac{\partial h}{\partial x}$ is only accurate for gentle to moderate slopes $S = \left|\frac{\partial h}{\partial x}\right|$.

For steep slopes flux rates increase when the slope approaches a critical value S_c: $q_s = -\frac{\rho\kappa \frac{\partial h}{\partial x}}{1 - \left(\left|\frac{\partial h}{\partial x}\right| S_c^{-1}\right)^2}$ leading to a non-linear hillslope diffusion.

Bedrock rivers

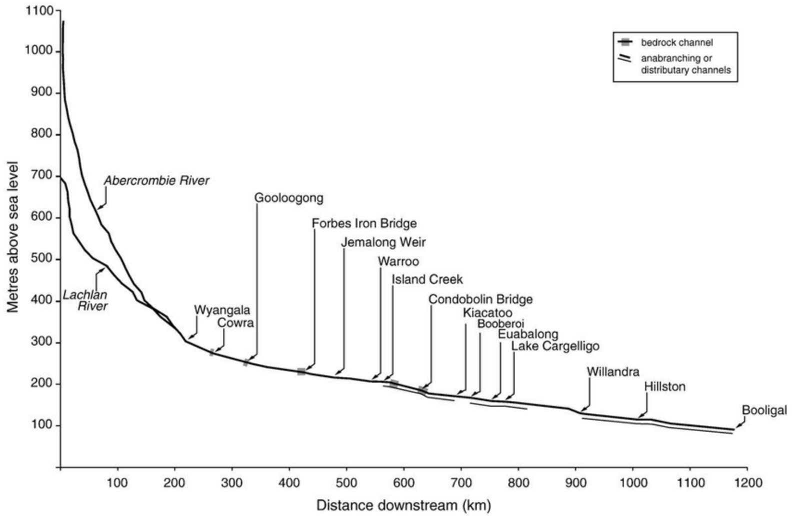
- Flow directly on the bedrock
- Detachment-limited
- Stream power $\omega = cQ_w S/B$
- Semi-empirical generalization for the erosion rate $E = CA^m S^n, A \sim Q_w$

Water discharge $\boldsymbol{Q_w}$, width $\boldsymbol{B}$, drainage area $\boldsymbol{A}$ and erodibility constants $\boldsymbol{c}, \boldsymbol{C} > 0$

- Evolution: $\frac{\partial h}{\partial t} = U - CA^m \left|\frac{\partial h}{\partial x}\right|^n$
- Steady state: $U = E$

Pennine Hills, England, © Jeff Warburton

https://eos.org/research-spotlights/how-do-rivers-flow-over-bedrock



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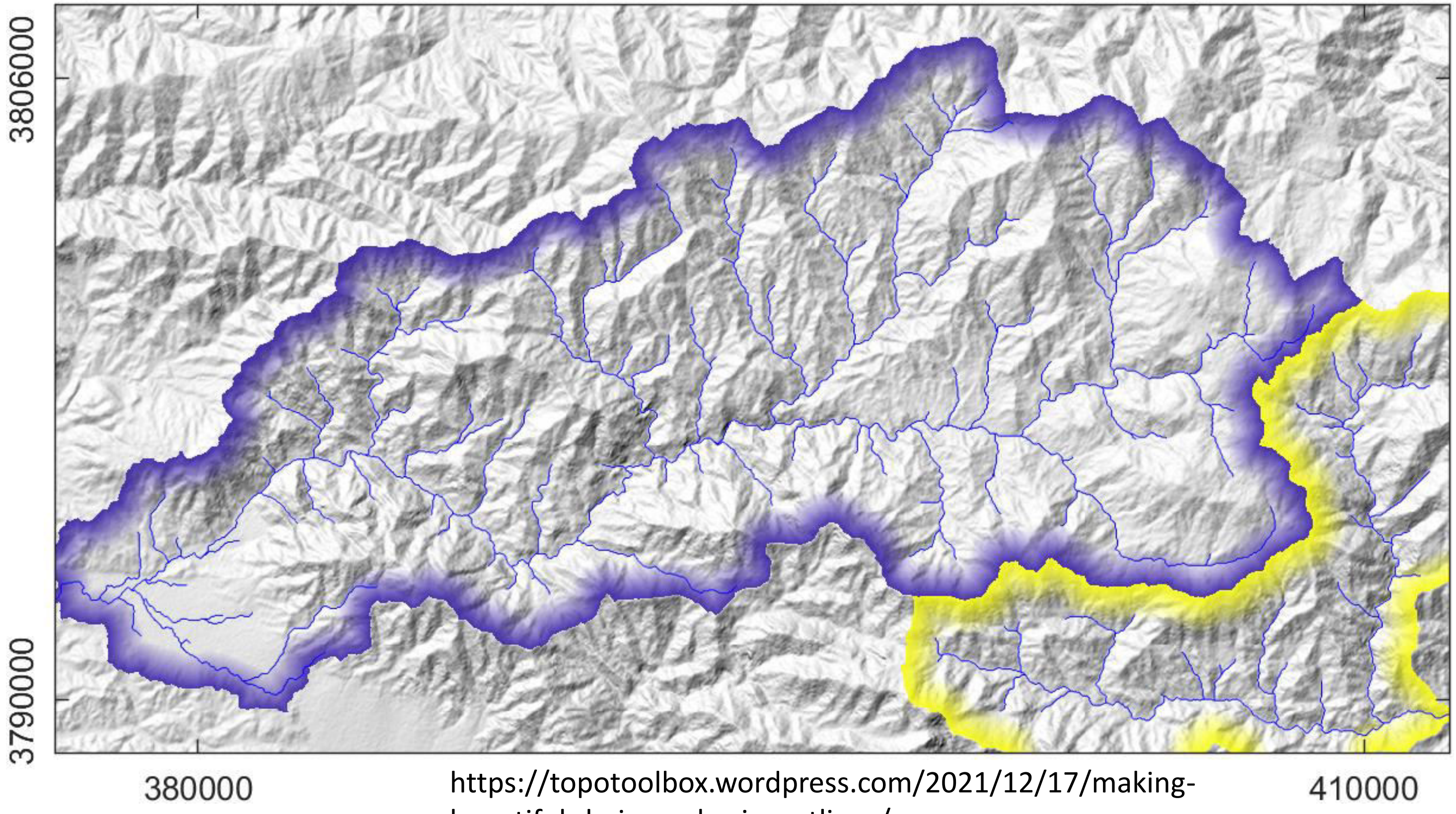
Bedrock rivers



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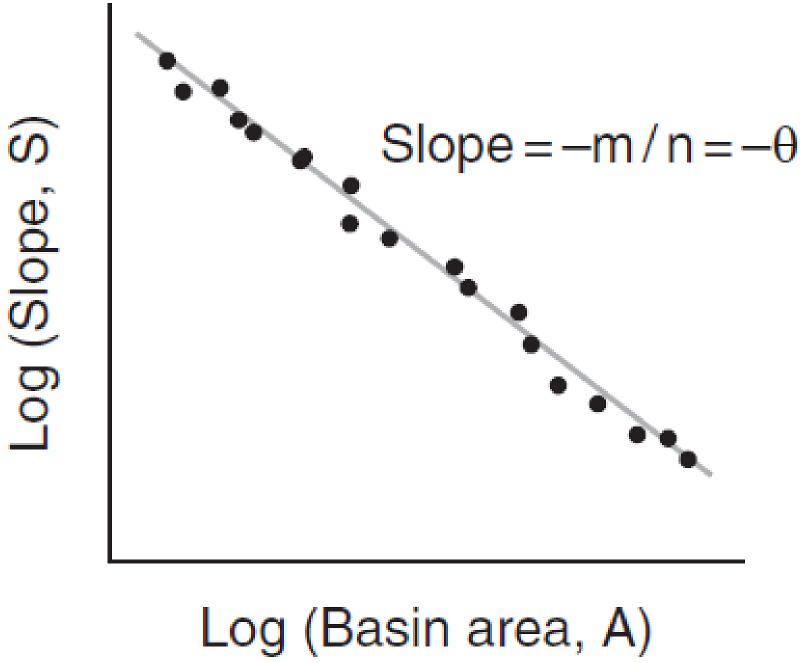


Anderson & Anderson: Geomorphology

In the steady state case, i.e. $E = U$,

$$S = \left(\frac{U}{C}\right)^{\frac{1}{n}} A^{-\frac{m}{n}}.$$

Drainage area A is used as proxy for water discharge Q_w.

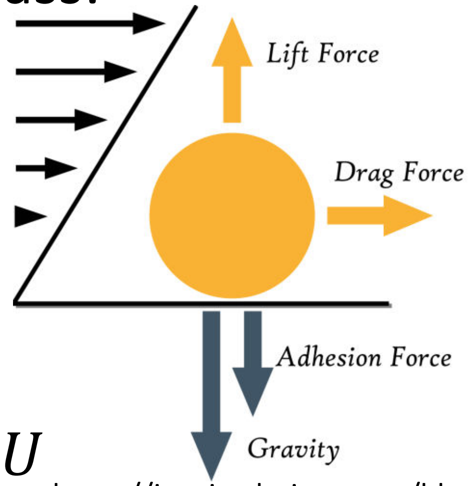
Alluvial rivers

- Flow on unconsolidated material transported by itself
- Transport-limited

Derivation of a long-profile evolution:

- **Exner equation** for conservation of mass:

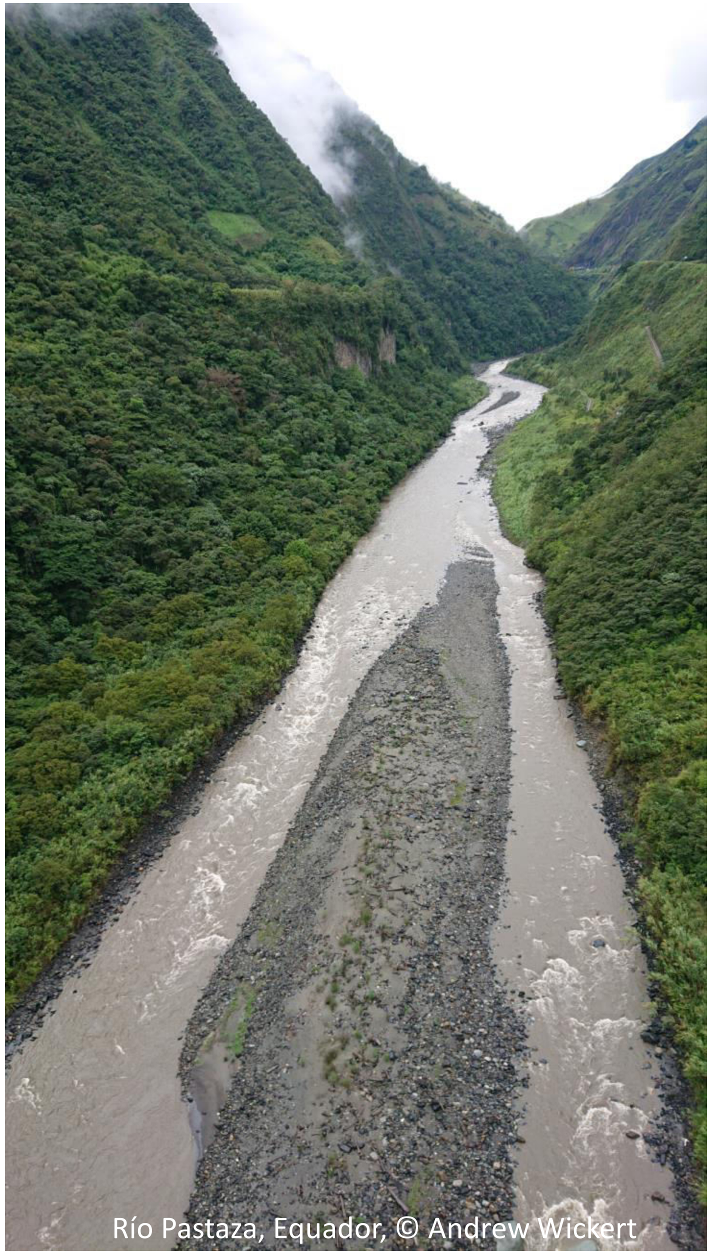
$$\frac{\partial h}{\partial t} = \frac{1}{B(1-\lambda_p)} \frac{\partial Q_s}{\partial x}$$



https://joaairsolutions.com/blog/understanding-force-balancing-model/

- $Q_s = C Q_w S^{\frac{7}{6}}$

$$\frac{\partial h}{\partial t} = \frac{C Q_w}{B} \left(\frac{\partial^2 h}{\partial x^2} + \frac{1}{Q_w} \frac{\partial Q_w}{\partial x} \frac{\partial h}{\partial x} \right) \left| \frac{\partial h}{\partial x} \right|^{\frac{1}{6}} + U$$



https://geomorphonline.github.io/fluvial/bedrock_alluvial/

Response times

Approach: e-folding

- Exponential curve fitting
- Difference to steady state shrinks to fraction of initial difference
- Essentially Ornstein-Uhlenbeck without noise

Example: Slope-evolution

$S(t) = S_{fin} + (S_{in} - S_{fin})e^{-\kappa t}$ with response time $\tau = \kappa^{-1}$ implies that

$$|S(\tau) - S_{fin}| = e^{-1}|S_{in} - S_{fin}|$$

Challenges:

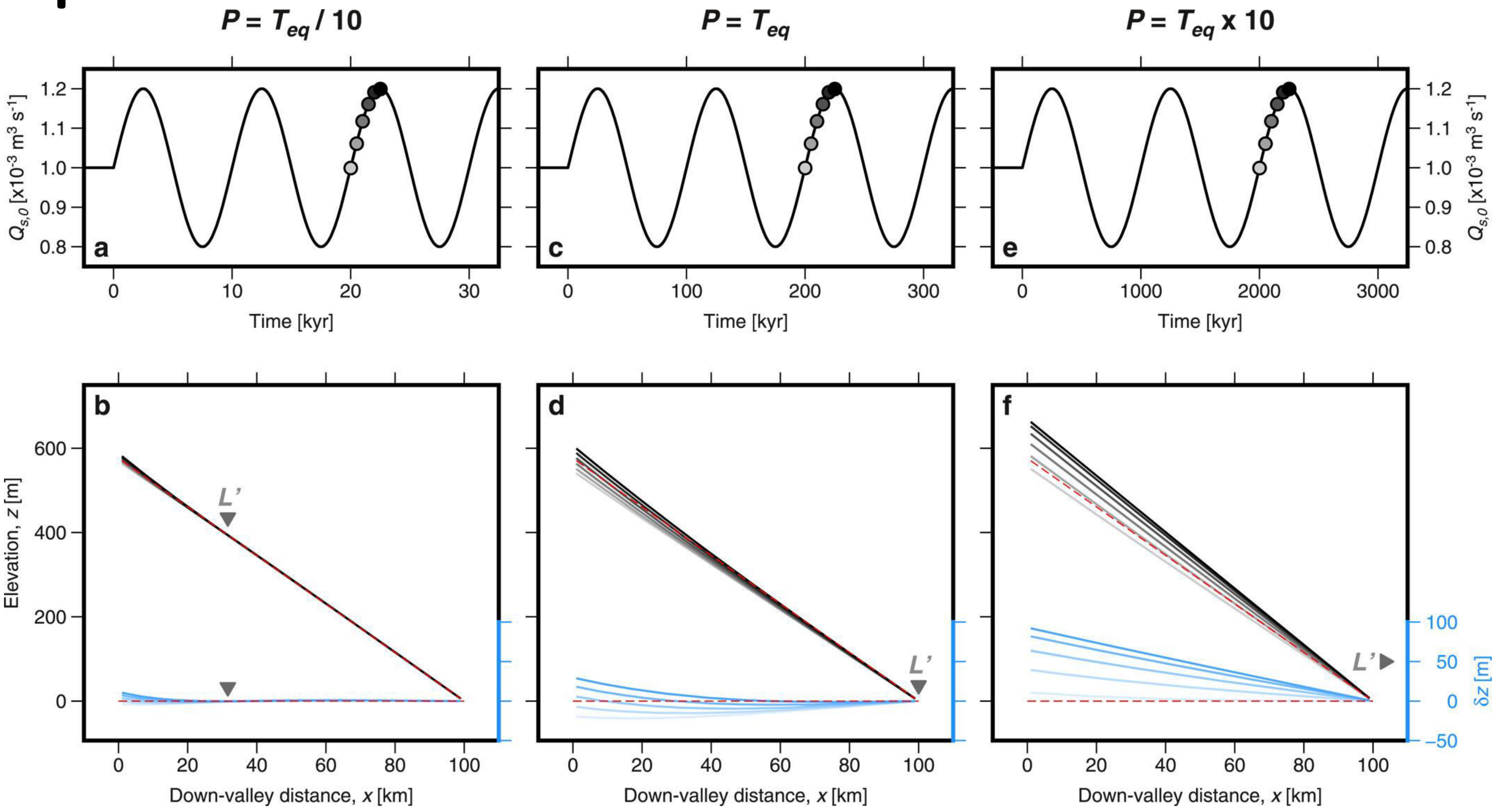
- Which quantity?
- Exponential approximation usually only good when close to the steady state (linear interpolation)
- Temporal/spatial scaling

Response times

Scaling

- Writing $\frac{\partial h}{\partial t}\left(\frac{H}{T}\right) = \kappa \frac{\partial^2 h}{\partial x^2}\left(\frac{H}{L^2}\right)$ in non-dimensional form yields $\frac{\partial h}{\partial t} = \left(\frac{\kappa T}{L^2}\right)\frac{\partial^2 h}{\partial x^2}$.
- Definition of dimensionless diffusivity $\left(\frac{\kappa T}{L^2}\right)$ and intrinsic scaled equilibrium time $T_{eq} = \frac{L^2}{\kappa}$.
- Definition of length scale $L' = \sqrt{P\kappa}$ for perturbation period P.

Response times



From McNab et al.

Analog experiments

What?

- Experiments in the lab under controlled conditions
- Often downscaled in size

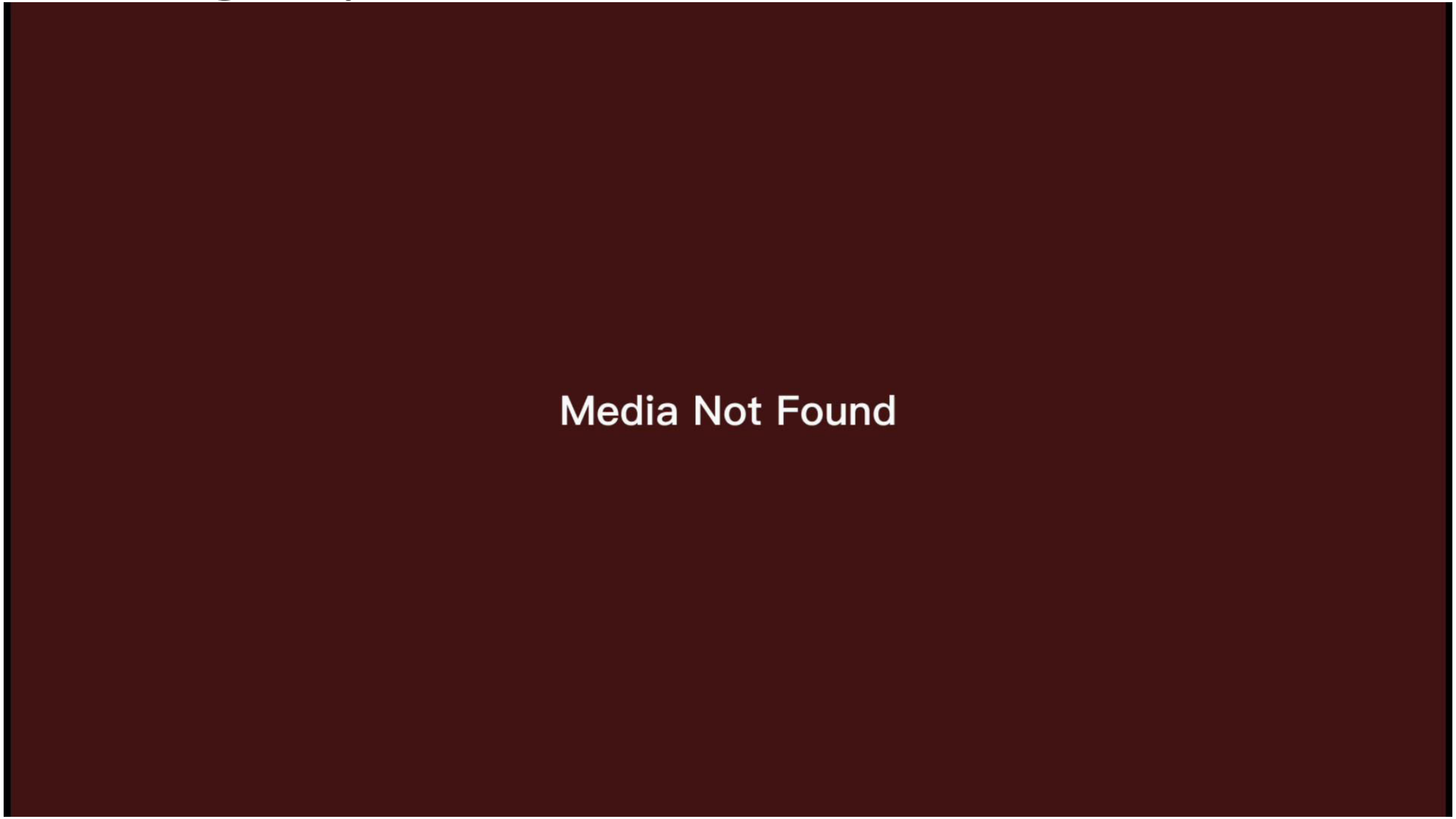
Why?

- Systematic testing
- High-resolution observations
- Learning about governing equations
- Hypothesis formulation

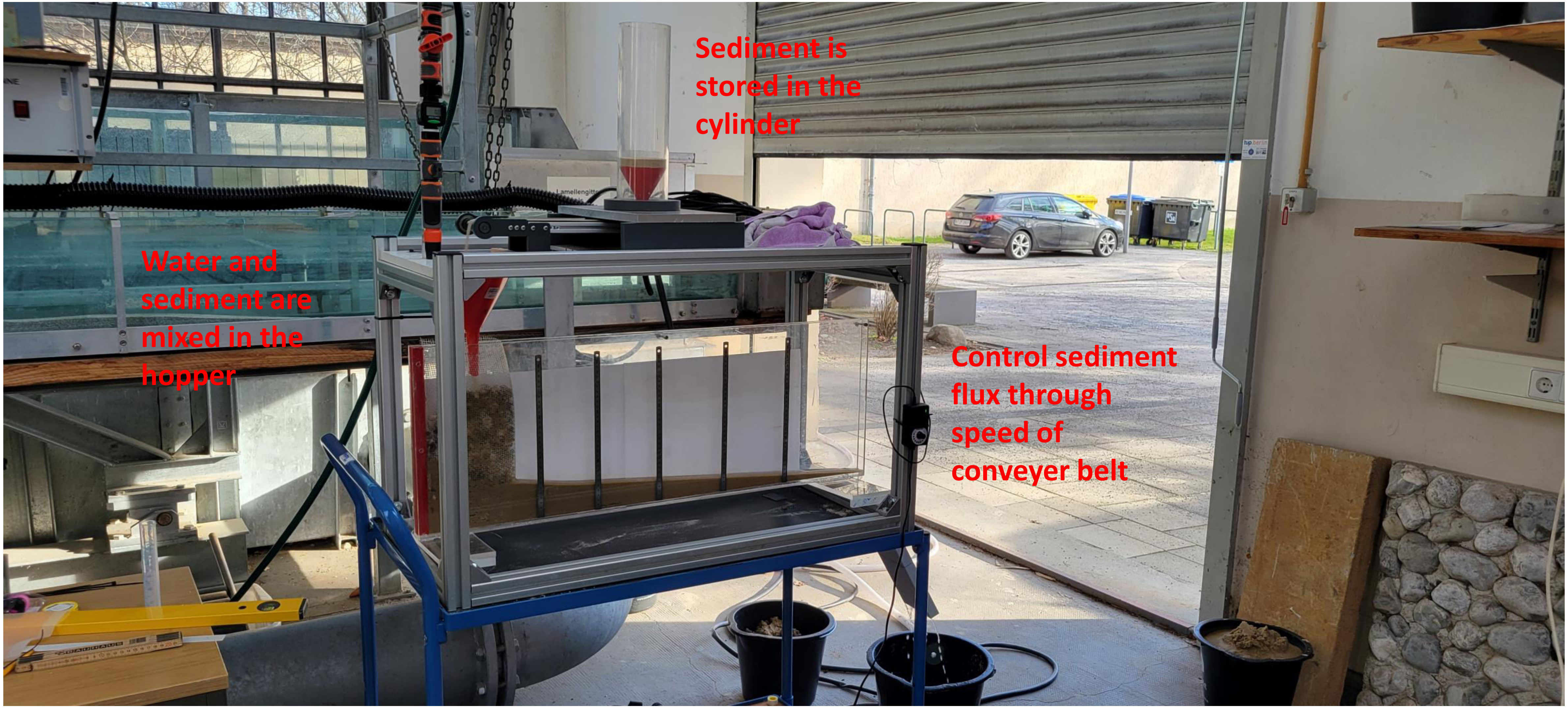
Potential issues:

- Incomplete reproduction of natural conditions
- Scaling results in non-similar dynamics (e.g. different Reynolds/Froude-number)

Analog experiments



Analog experiments



Analog experiments

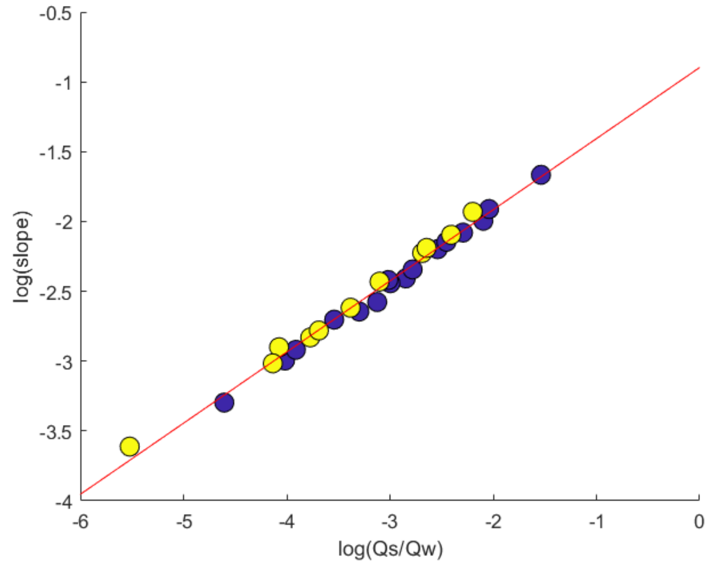
Question: Different response times in Aggradation vs Incision?

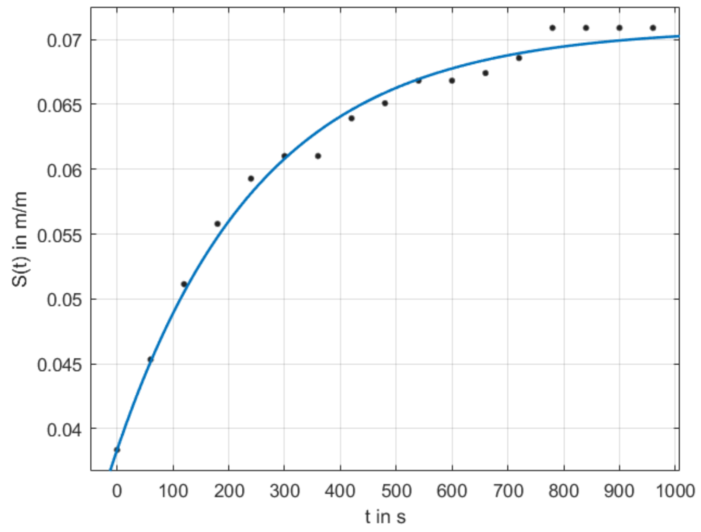
Data: $S(t)$ for different values of Q_s and Q_w

Results:

- Power law relationship $\left(\frac{Q_s}{Q_w}\right)^{\alpha} \sim S_{fin}$
- Fairly well e-folding fit: $S(t) = S_{fin} + (S_{in} - S_{fin})e^{-\kappa t}$

Conjecture: Response time scales with $|S_{in} - S_{fin}|$, Q_w and Q_s





Challenges

- Insufficient resolution
- Non-observables
- Empirical laws
- Observational noise
- Complexity of models

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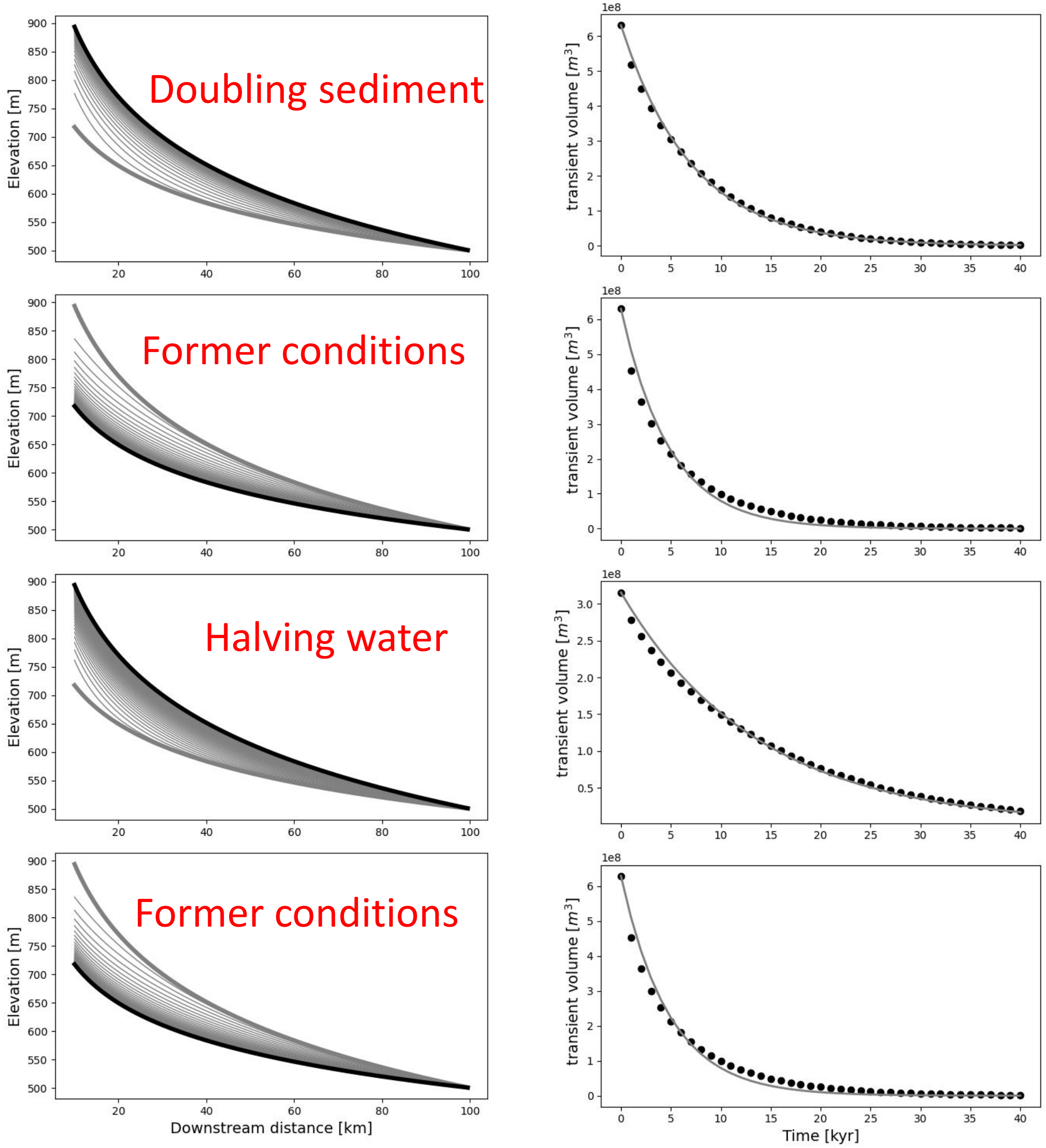
Alluvial rivers

Simulation results

- Doubling sediment flux Q_s and returning to former conditions results in similar response times of roughly **7.1ky** and **4.8ky**, resp.
- Halving water discharge Q_w and returning to former conditions results in roughly **13.6ky** and **4.8ky**.
- Similar slopes are formed, yet response times differ. Q_w works as catalysts.



Further examples of (S)PDEs in Geosciences

- Climate models
- Weather prediction
- Element concentration/distribution
- Hydrology, e.g. groundwater flow, flow in open channel (from Navier-Stokes), contamination…